Appendix to 'A bifurcation threshold for contact-induced language change':

Mathematical derivations

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A Learning: definitions

Following Bush and Mosteller (1955), suppose the learner has n possible actions (here, n possible grammars) $G_1, \ldots, G_n$ at their disposal and uses the ith action with probability p_i, so that the learner's knowledge is represented by the probability vector $\mathbf{p} = (p_1, \ldots, p_n)$. Once the learner has chosen an action and acted on the environment, the latter responds with one of m possible responses $R_1, \ldots, R_m$. The probability of the jth response occurring, given that the learner chose the ith action, will be denoted ω_{ij}. The set of these probabilities defines the (stationary random) learning environment; we require, of course, that $\sum_j \omega_{ij} = 1$ for all i. Suppose the learner chose G_i and that the environment responded with R_j. Having observed this, the learner adjusts the vector $\mathbf{p}$ by applying an operator f_{ij} which, in the general case, is only required to be some mapping $f_{ij} : \Delta^{n-1} \to \Delta^{n-1}$ from the simplex

$$\Delta^{n-1} = \{\mathbf{p} = (p_1, \ldots, p_n) \in [0, 1]^n : \textstyle\sum_i p_i = 1\} \tag{1}$$

to itself. The process is then repeated: the next time, the learner chooses an action by drawing from the distribution $f_{ij}(\mathbf{p})$, and some $f_{k\ell}$ is applied to this vector to yield $f_{k\ell}(f_{ij}(\mathbf{p}))$, and so on.

Many of the formal properties of this general framework are understood in detail for various choices of operators f_{ij}, both for stationary (constant ω_{ij}) and non-stationary (time-dependent ω_{ij}) environments (Bush & Mosteller, 1955; Narendra & Thathachar, 1989). In what follows, I will utilize a special case in which the learner has two actions, the environment is stationary and has two responses, and the operators f_{ij} are linear. The assumptions of stationarity and linearity facilitate characterization of the asymptotic distribution of a population of learners and turn out to give rise to tractable (although nonlinear) dynamics across generations of learners. The restriction to $n = 2$ actions is not necessary, but is made to keep the presentation manageable and also in view of the empirical application, which concerns this special case. The two possible environmental responses will be interpreted as reward and punishment, in a sense to be made precise later.

Thus let $\mathbf{p} = (p_1, p_2) = (p, 1-p)$. From now on, I will interpret the variables in a linguistic setting and identify $p_1 = p$ with the probability of grammar G_1 and $p_2 = 1 - p$ with the probability of grammar G_2. I assume in all that follows that grammar G_1 incurs some amount of L2-difficulty, while G_2 is not subject to such an inherent bias. In other words, adult L2 learners are expected to struggle more in acquiring G_1 than in acquiring G_2.

Assuming the learner is making a binary choice between two grammars and that the environment signals two different responses only, four operators f_{ij} need to be specified. Assuming further that these operators are linear in $\mathbf{p} = (p_1, p_2) = (p, 1 - p)$, we have

$$f_{ij}(\mathbf{p}) = M^{(ij)}\mathbf{p} = \begin{bmatrix} m_{11}^{(ij)} & m_{12}^{(ij)} \\ m_{21}^{(ij)} & m_{22}^{(ij)} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} m_{11}^{(ij)} p_1 + m_{12}^{(ij)} p_2 \\ m_{21}^{(ij)} p_1 + m_{22}^{(ij)} p_2 \end{bmatrix} \tag{2}$$

for some matrix $M^{(ij)}$ with suitable constants $m_{k\ell}^{(ij)}$ chosen so that $M^{(ij)}\mathbf{p} \in \Delta^1$. In this one-dimensional case we can, of course, drop p_2 and work with functions that operate on the scalar $p_1 = p$:

$$f_{ij}(p) = m_{11}^{(ij)}p + m_{12}^{(ij)}(1-p) = (m_{11}^{(ij)} - m_{12}^{(ij)})p + m_{12}^{(ij)} \tag{3}$$

or, what amounts to the same thing,

$$f_{ij}(p) = a_{ij}p + b_{ij} \tag{4}$$

for constants a_{ij}, b_{ij} $(i, j = 1, 2)$. These operators are thus affine in p.

A classical choice is to interpret response R_1 as reward and response R_2 as punishment, and to set

$$\left.\begin{aligned} f_{11}(p) &= (1-\gamma)p + \gamma \\ f_{12}(p) &= (1-\gamma)p \\ f_{21}(p) &= (1-\gamma)p \\ f_{22}(p) &= (1-\gamma)p + \gamma \end{aligned}\right\} \quad \text{i.e.} \quad \left.\begin{aligned} f_{11}(p) &= p + \gamma(1-p) \\ f_{12}(p) &= p - \gamma p \\ f_{21}(p) &= p - \gamma p \\ f_{22}(p) &= p + \gamma(1-p) \end{aligned}\right\} \tag{5}$$

for constant $0 < \gamma < 1$. In other words, the value of p is augmented whenever grammar G_1 is rewarded or G_2 is punished, and decreased otherwise, as is evident from the representation on the right in (5). The parameter γ, which governs how large or small modifications to p the learner makes in response to the environment's responses, can be interpreted as a learning rate.

The operators in (5) constitute the one-dimensional linear reward–penalty scheme of Bush and Mosteller (1955), first applied to linguistic problems by Yang (2000). To extend this model to cater for adult L2 acquisition, I now assume that adult L2 learners employ the same general learning strategy but are, additionally, subject to a bias which discounts grammars that are L2-difficult. As mentioned above, I take G_1 to refer to the grammar that incurs L2-difficulty, and assume G_2 not to be targeted by a similar bias. A simple extension of (5) is then the following, for secondary learning rate parameter δ:

$$\left.\begin{aligned} f_{11}(p) &= (1-\gamma-\delta)p + \gamma \\ f_{12}(p) &= (1-\gamma-\delta)p \\ f_{21}(p) &= (1-\gamma-\delta)p \\ f_{22}(p) &= (1-\gamma-\delta)p + \gamma \end{aligned}\right\} \quad \text{i.e.} \quad \left.\begin{aligned} f_{11}(p) &= p + \gamma(1-p) - \delta p \\ f_{12}(p) &= p - \gamma p - \delta p \\ f_{21}(p) &= p - \gamma p - \delta p \\ f_{22}(p) &= p + \gamma(1-p) - \delta p \end{aligned}\right\}. \tag{6}$$

As is evident from the forms on the right, the additional term $-\delta p$ (with positive δ) constitutes a negative bias experienced by grammar G_1, regardless of the environment's response. We require $0 \leq \delta \leq 1 - \gamma$ to guarantee p always remains in the interval $[0, 1]$.

B Learning: asymptotic results

Learning, under this operationalization, is a stochastic process. In particular, it is (in practice) impossible to predict the exact evolution of p given an initial state p_0. However, an explicit recursive solution exists for all moments of the distribution of p:

Lemma 1 (Bush & Mosteller 1955: 98)**.** *For any linear operators of the form (4) in a stationary random environment, the following recursion holds for the mth moment of p, $\langle p^m \rangle$:*

$$\langle p^m \rangle_{n+1} = \sum_{k=0}^{m} \binom{m}{k} \left(\Omega'_{m,k} \langle p^k \rangle_n + (\Omega_{m,k} - \Omega'_{m,k}) \langle p^{k+1} \rangle_n \right), \tag{7}$$

where

$$\left.\begin{aligned} \Omega_{m,k} &= a_{11}^k b_{11}^{m-k} \omega_{11} + a_{12}^k b_{12}^{m-k} \omega_{12} \\ \Omega'_{m,k} &= a_{21}^k b_{21}^{m-k} \omega_{21} + a_{22}^k b_{22}^{m-k} \omega_{22} \end{aligned}\right\}. \tag{8}$$

It will be useful to define the following averages (for $i = 1, 2$):

$$\left.\begin{aligned} \overline{a}_i &= a_{i1}\omega_{i1} + a_{i2}\omega_{i2} \\ \overline{b}_i &= b_{i1}\omega_{i1} + b_{i2}\omega_{i2} \\ \overline{\overline{a}}_i &= a_{i1}^2\omega_{i1} + a_{i2}^2\omega_{i2} \\ \overline{\overline{b}}_i &= b_{i1}^2\omega_{i1} + b_{i2}^2\omega_{i2} \\ \overline{ab}_i &= a_{i1}b_{i1}\omega_{i1} + a_{i2}b_{i2}\omega_{i2} \end{aligned}\right\}. \tag{9}$$

It is then a straightforward algebraic exercise to derive the following results concerning the first two raw moments from Lemma 1:

$$\left.\begin{aligned} \langle p\rangle_{n+1} &= \overline{b}_2 + (\overline{b}_1 - \overline{b}_2 + \overline{a}_2)\langle p\rangle_n + (\overline{a}_1 - \overline{a}_2)\langle p^2\rangle_n \\ \langle p^2\rangle_{n+1} &= \overline{\overline{b}} + (\overline{\overline{b}}_1 - \overline{\overline{b}}_2 + 2\overline{ab}_2)\langle p\rangle_n + (2\overline{ab}_1 - 2\overline{ab}_2 + \overline{\overline{a}}_2)\langle p^2\rangle_n + (\overline{\overline{a}}_1 - \overline{\overline{a}}_2)\langle p^3\rangle_n \end{aligned}\right\}. \tag{10}$$

Thus, in the general case, the mth moment depends on the $(m+1)$th moment. This problematic upward dependence disappears, however, if $\overline{a}_1 - \overline{a}_2 = 0$ and $\overline{\overline{a}}_1 - \overline{\overline{a}}_2 = 0$. Using the fact that $\omega_{i1} + \omega_{i2} = 1$, it is easy to check that this is the case if $a_{ij} = a$ for some common a, that is to say, if the slopes of the four affine operators are identical. This is obviously the case with both the classical linear reward–penalty scheme (5) as well as its extension to L2 learning (6). For the mean, we then have

$$\langle p\rangle_{n+1} = C_0 + C_1\langle p\rangle_n \tag{11}$$

with $C_0 = \overline{b}_2$ and $C_1 = \overline{b}_1 - \overline{b}_2 + \overline{a}_2$, a simple linear difference equation with solution

$$\langle p\rangle_n = C_1^n\langle p\rangle_0 + (1 - C_1^n)\langle p\rangle_\infty \tag{12}$$

where

$$\langle p\rangle_\infty = \frac{C_0}{1 - C_1} \tag{13}$$

is the limit at $n \to \infty$ as long as $|C_1| < 1$. The latter inequality is easy to verify for algorithms (5) and (6) as long as the environment satisfies $0 < \omega_{ii} < 1$, which we can assume without loss of generality.

Lemma 2. *Assume the learning environment satisfies* $0 < \omega_{ii} < 1$ *for* $i = 1, 2$. *Then* $|C_1| < 1$ *for both* (5) *and* (6).

Proof. The constant C_1 has been defined as

$$C_1 = \overline{b}_1 - \overline{b}_2 + \overline{a}_2. \tag{14}$$

For algorithm (6) this becomes, using the definitions in (9),

$$C_1 = (\omega_{11} - \omega_{22})\gamma - \gamma + 1 - \delta. \tag{15}$$

The learning rate parameters are assumed to satisfy $0 < \gamma < 1$ and $0 \leq \delta \leq 1 - \gamma$, which implies $1 - \delta \geq \gamma$. Hence

$$C_1 \geq (\omega_{11} - \omega_{22})\gamma. \tag{16}$$

Since ω_{11} and ω_{22} are probabilities and we furthermore assume they belong to the open interval $]0, 1[$, their difference satisfies $-1 < \omega_{11} - \omega_{22} < 1$. Hence

$$C_1 > -\gamma > -1. \tag{17}$$

On the other hand, since $\delta \geq 0$,

$$C_1 \leq (\omega_{11} - \omega_{22})\gamma - \gamma + 1 < \gamma - \gamma + 1 = 1. \tag{18}$$

All in all, $|C_1| < 1$. □

Taking the constants a_{ij}, b_{ij} from (6) we then have, after simplifying all the factors,

$$\langle p \rangle_\infty = \frac{\omega_{22}}{\omega_{12} + \omega_{22} + d}, \tag{19}$$

where $d = \delta/\gamma$. Given that response R_2 was identified as punishment, the parameters ω_{12} and ω_{22} here refer to the probability of grammars G_1 and G_2 being punished, respectively. In line with previous work (Yang, 2000), I will call these the *penalty probabilities* associated with the two grammars, and will write $\pi_1 = \omega_{12}$ and $\pi_2 = \omega_{22}$ in what follows for simplicity. We have thus shown:

Proposition 1. *For algorithm* (6)*, the expected value of* p*, the probability of use of grammar* G_1*, after an infinity of learning iterations is*

$$\langle p \rangle_\infty = \frac{\pi_2}{\pi_1 + \pi_2 + d}, \tag{20}$$

where π_1 *and* π_2 *are the penalty probabilities of the two grammars and* $d = \delta/\gamma$ *supplies the relative L2-difficulty of grammar* G_1*. The asymptotic expectation for algorithm* (5) *is obtained by setting* $\delta = 0$ *and thus* $d = 0$*.*

The second raw moment, in turn, evolves as

$$\langle p^2 \rangle_{n+1} = D_0 + D_1 \langle p^2 \rangle_n + D_2 \langle p \rangle_n \tag{21}$$

with $D_0 = \overline{\overline{b}}_2$, $D_1 = 2\overline{ab}_1 - 2\overline{ab}_2 + \overline{\overline{a}}_2$ and $D_2 = \overline{\overline{b}}_1 - \overline{\overline{b}}_2 + 2\overline{ab}_2$. Plugging the solution for the mean (12) in this equation we have

$$\langle p^2 \rangle_{n+1} = D_0 + D_1 \langle p^2 \rangle_n + D_2 (\langle p \rangle_0 - \langle p \rangle_\infty) C_1^n + D_2 \langle p \rangle_\infty \tag{22}$$

or, in other words,

$$\langle p^2 \rangle_{n+1} = E_0 + D_1 \langle p^2 \rangle_n + E_1 C_1^n, \tag{23}$$

where $E_0 = D_0 + D_2 \langle p \rangle_\infty$ and $E_1 = D_2 (\langle p \rangle_0 - \langle p \rangle_\infty)$. For large n, we ignore the term $E_1 C_1^n$ since $|C_1| < 1$. Hence as $n \to \infty$, the second raw moment tends to the limit

$$\langle p^2 \rangle_\infty = \frac{E_0}{1 - D_1} = \frac{D_0 + D_2 \langle p \rangle_\infty}{1 - D_1}. \tag{24}$$

It can further be shown that, for algorithms (5) and (6) and for fixed d, $\langle p^2 \rangle_\infty \to \langle p \rangle_\infty^2$ as $\gamma \to 0$, meaning that the limiting variance of p, $V[p]_\infty = \langle p^2 \rangle_\infty - \langle p \rangle_\infty^2$, converges to zero:

Proposition 2. *For both* (5) *and* (6)*, the variance of* p *in the limit* $n \to \infty$ *can be made arbitrarily small by assuming a sufficiently small learning rate.*

Proof. We show the result for algorithm (6); the statement for algorithm (5) follows as the special case $\delta = 0$. Recall that algorithm (6) consists of the statement that the constants of the linear operators satisfy $a_{ij} = a = 1 - \gamma - \delta$ (for $i, j = 1, 2$) and $b_{11} = b_{22} = \gamma$, $b_{12} = b_{21} = 0$, where $0 < \gamma \leq 1$ and $0 \leq \delta \leq 1 - \gamma$.

The difference in response probabilities $\omega_{22} - \omega_{11}$ will recur often in the following calculations; let $\omega = \omega_{22} - \omega_{11}$ for convenience.

Let $d = \delta/\gamma$ and assume d is fixed, so that as $\gamma \to 0$, also $\delta \to 0$. Substituting the constants $a_{ij} = a$ and b_{ij} in the definitions (9) yields

$$\left.\begin{aligned} D_0 &= \overline{\overline{b}}_2 = \gamma^2 \omega_{22} \\ D_1 &= 2\overline{ab}_1 - 2\overline{ab}_2 + \overline{\overline{a}}_2 = a^2 - 2a\gamma\omega \\ D_2 &= \overline{\overline{b}}_1 - \overline{\overline{b}}_2 + 2\overline{ab}_2 = 2a\gamma\omega_{22} - \gamma^2\omega \end{aligned}\right\}. \tag{25}$$

Moreover,

$$1 - a^2 = 1 - (1 - \gamma - \delta)^2 = 2\gamma + 2\delta - \gamma^2 - 2\gamma\delta - \delta^2 = \gamma\left(2 + 2\frac{\delta}{\gamma} - \gamma - 2\delta - \frac{\delta^2}{\gamma}\right), \quad (26)$$

in other words

$$1 - a^2 = \gamma(2 + 2d - \gamma - 2\delta - d\delta). \quad (27)$$

It now follows that

$$1 - D_1 = 1 - a^2 + 2a\gamma\omega = \gamma(2 + 2d - \gamma - 2\delta - d\delta) + 2a\gamma\omega. \quad (28)$$

On the other hand

$$\langle p^2 \rangle_\infty = \frac{D_0 + D_2 \langle p \rangle_\infty}{1 - D_1}, \quad (29)$$

in other words,

$$\langle p^2 \rangle_\infty = \frac{\gamma^2 \omega_{22} + (\gamma^2 \omega + 2a\gamma\omega_{22})\langle p \rangle_\infty}{\gamma(2 + 2d - \gamma - 2\delta - d\delta) + 2a\gamma\omega} = \frac{\gamma\omega_{22} + (\gamma\omega + 2a\omega_{22})\langle p \rangle_\infty}{2 + 2d - \gamma - 2\delta - d\delta + 2a\omega}. \quad (30)$$

As $\gamma \to 0$, $\delta \to 0$ and $a \to 1$. Hence

$$\lim_{\gamma \to 0} \langle p^2 \rangle_\infty = \frac{2\omega_{22} \langle p \rangle_\infty}{2 + 2d + 2\omega} = \frac{\omega_{22}}{1 + \omega + d} \langle p \rangle_\infty. \quad (31)$$

On the other hand,

$$1 + \omega = 1 - \omega_{11} + \omega_{22} = \omega_{21} + \omega_{22}. \quad (32)$$

Recalling the notational convention $\pi_1 = \omega_{21}$ and $\pi_2 = \omega_{22}$, we now have, with the help of Proposition 1,

$$\lim_{\gamma \to 0} \langle p^2 \rangle_\infty = \frac{\pi_2}{\pi_1 + \pi_2 + d} \langle p \rangle_\infty = \langle p \rangle_\infty \langle p \rangle_\infty = \langle p \rangle_\infty^2 \quad (33)$$

as desired. □

To recap, a population of learners employing either the linear reward–penalty scheme (5) or its L2 extension (6) will tend to a mean value of p in the limit of large learning iterations which is given by Proposition 1. Moreover, if learning is slow, so that the learning rates γ and δ have small values, variability between learners in this population will be small. To be exact, that variability vanishes as γ and δ tend to zero.

C Population dynamics

We now concentrate on a learning environment characterized by the following penalty probabilities (see the main paper for motivation):

$$\left.\begin{aligned} \pi_1 &= (1 - \sigma)\alpha_2(1 - p) + \sigma\alpha_2(1 - q) \\ \pi_2 &= (1 - \sigma)\alpha_1 p + \sigma\alpha_1 q \end{aligned}\right\}, \quad (34)$$

where p and q are the probabilities of grammar G_1 in the L1 and L2 speaker populations, respectively, σ is the fraction of L2 speakers in the overall population, and α_1 and α_2 are the grammatical advantages of G_1 and G_2.

Making use of the asymptotic results from the previous section, we have the following general ansatz for inter-generational difference equations:

$$\left.\begin{aligned} p_{n+1} - p_n &= \langle p_n \rangle_\infty - p_n \\ q_{n+1} - q_n &= \langle q_n \rangle_\infty - q_n \end{aligned}\right\}. \quad (35)$$

We may, without loss of generality, study the continuous-time limit

$$\left.\begin{aligned}\dot{p} &= \langle p\rangle_\infty - p\\ \dot{q} &= \langle q\rangle_\infty - q\end{aligned}\right\} \tag{36}$$

instead. With Proposition 1, this becomes

$$\left.\begin{aligned}\dot{p} &= \frac{\pi_2 - (\pi_1+\pi_2)p}{\pi_1+\pi_2} = \frac{\pi_2(1-p) - \pi_1 p}{\pi_1+\pi_2}\\ \dot{q} &= \frac{\pi_2 - (\pi_1+\pi_2+d)q}{\pi_1+\pi_2+d} = \frac{\pi_2(1-q) - \pi_1 q - dq}{\pi_1+\pi_2+d}\end{aligned}\right\}. \tag{37}$$

The denominators are strictly positive as long as $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, which is the case of interest here. They therefore do not contribute to the system's equilibria, and we may drop them without loss of generality. Doing this, filling in the penalties from (34), and adopting the notational shorthand $\widetilde{x} = 1 - x$ for any real x, we now have

$$\left.\begin{aligned}\dot{p} &= \alpha_1(\widetilde{\sigma}p + \sigma q)\widetilde{p} - \alpha_2(\widetilde{\sigma}\widetilde{p} + \sigma\widetilde{q})p\\ \dot{q} &= \alpha_1(\widetilde{\sigma}p + \sigma q)\widetilde{q} - \alpha_2(\widetilde{\sigma}\widetilde{p} + \sigma\widetilde{q})q - dq\end{aligned}\right\}. \tag{38}$$

Division of the right hand sides by α_2, again without loss of generality, finally yields

$$\left.\begin{aligned}\dot{p} &= \alpha(\widetilde{\sigma}p + \sigma q)\widetilde{p} - (\widetilde{\sigma}\widetilde{p} + \sigma\widetilde{q})p\\ \dot{q} &= \alpha(\widetilde{\sigma}p + \sigma q)\widetilde{q} - (\widetilde{\sigma}\widetilde{p} + \sigma\widetilde{q} + D)q\end{aligned}\right\}, \tag{39}$$

where $\alpha = \alpha_1/\alpha_2$ gives the ratio of the grammatical advantages and $D = d/\alpha_2$ represents the L2-difficulty of grammar G_1 scaled by the advantage of grammar G_2. It is easy to check (by examining the signs of $\dot{p}$ and $\dot{q}$ at the four sides of $[0,1]^2$) that this system is well-defined, in the sense that the unit square $[0,1]^2$ is forward-invariant under the dynamics.

Examination of (39) quickly shows that the origin $(p,q) = (0,0)$ is always an equilibrium of this system, for any selection of parameter values α, D and σ. A further, non-origin equilibrium may exist in $[0,1]^2$ depending on the combination of parameter values.

Proposition 3. *The system* (39) *has either one or two equilibria, for any combination of values of the parameters α, D and σ. The origin $(p,q) = (0,0)$ is always an equilibrium.*

Proof. The two nullclines of (39), i.e. the sets

$$N_p = \{(p,q) \in [0,1]^2 : \dot{p} = 0\} \tag{40}$$

and

$$N_q = \{(p,q) \in [0,1]^2 : \dot{q} = 0\} \tag{41}$$

are quadratic in p and q and define hyperbolas in the pq-plane in the general case. (An exception is the trivial case $\alpha = 1$, in which N_p and N_q reduce to straight lines intersecting at the origin.) In general, these hyperbolas will not have their centres at the origin, nor will their axes of symmetry be parallel to the coordinate axes. The hyperbolas may also, in the general case, intersect in up to four points in the real plane. Here, we show that at least one and at most two of those intersections occur in $[0,1]^2$.

Performing the relevant substitutions in (39), it is quick to verify that the origin $(0,0)$ always belongs to both nullclines, and that $(1,1)$ always belongs to N_p. To show that N_p and N_q intersect in no more than two points in $[0,1]^2$, we examine the hyperbola N_p in detail. Setting the first equation in (39) to zero, multiplying all terms out and rearranging, we have the canonical second-degree equation

$$A_{pp}p^2 + 2A_{pq}pq + A_{qq}q^2 + B_p p + B_q q + C = 0 \tag{42}$$

with

$$\left.\begin{aligned}A_{pp} &= \widetilde{\alpha}\widetilde{\sigma}\\ 2A_{pq} &= \widetilde{\alpha}\sigma\\ A_{qq} &= 0\\ B_p &= \alpha\widetilde{\sigma} - 1\\ B_q &= \alpha\sigma\\ C &= 0\end{aligned}\right\}. \tag{43}$$

The idea now is to show that the centre of the hyperbola, (p_c, q_c), always satisfies either (i) $p_c < 0$ and $q_c > 1$, or (ii) $p_c > 1$ and $q_c < 0$, and that therefore one of its branches never intersects $[0,1]^2$. Using a translation of the coordinate system (see Kelly & Straus, 1968, pp. 246–247), the centre is found to be at

$$(p_c, q_c) = \left(\frac{B_q A_{pq} - B_p A_{qq}}{2\Delta}, \frac{B_p A_{pq} - B_q A_{pp}}{2\Delta}\right), \tag{44}$$

where Δ is the discriminant

$$\Delta = A_{pp}A_{qq} - A_{pq}^2. \tag{45}$$

With the coefficients (43), we find

$$(p_c, q_c) = \left(\frac{-\alpha}{\widetilde{\alpha}}, \frac{\alpha\widetilde{\sigma} + 1}{\widetilde{\alpha}\sigma}\right). \tag{46}$$

Now, it is easy to check that, whenever $0 < \alpha < 1$, we have $p_c < 0$ and $q_c > 1$, and on the other hand that, when $\alpha > 1$, the conditions $p_c > 1$ and $q_c < 0$ obtain. Thus one of the branches of N_p never touches the unit square $[0,1]^2$. On the other hand, as the other branch of N_p always passes through both $(0,0)$ and $(1,1)$, and as N_q always passes through $(0,0)$, it follows that N_p and N_q can intersect in at most one other point in $[0,1]^2$ in addition to the origin. In other words, the system (39) has either one or two equilibria in $[0,1]^2$. □

When the non-origin equilibrium exists, it is tedious to solve (39) for it explicitly in the general case. However, to understand the qualitative dynamics it suffices to study the conditions under which the equilibrium at the origin reverses stability, giving rise to the second equilibrium. To do this, we linearize about the origin, i.e. inspect the eigenvalues of the system's Jacobian matrix at that point. This is

$$J(p,q) = \begin{pmatrix} 2\widetilde{\alpha}\widetilde{\sigma}p + \widetilde{\alpha}\sigma q + \alpha\widetilde{\sigma} - 1 & \widetilde{\alpha}\sigma p + \alpha\sigma \\ \widetilde{\alpha}\widetilde{\sigma}q + \alpha\widetilde{\sigma} & \widetilde{\alpha}\widetilde{\sigma}p + 2\widetilde{\alpha}\sigma q + \alpha\sigma - D - 1 \end{pmatrix}. \tag{47}$$

Evaluated at the origin, the Jacobian reduces to

$$J(0,0) = \begin{pmatrix} \alpha\widetilde{\sigma} - 1 & \alpha\sigma \\ \alpha\widetilde{\sigma} & \alpha\sigma - D - 1 \end{pmatrix}. \tag{48}$$

An eigenvalue λ must satisfy

$$\begin{vmatrix} \alpha\widetilde{\sigma} - 1 - \lambda & \alpha\sigma \\ \alpha\widetilde{\sigma} & \alpha\sigma - D - 1 - \lambda \end{vmatrix} = 0, \tag{49}$$

which upon computation of the determinant yields the characteristic polynomial

$$\lambda^2 + (D - \alpha + 2)\lambda + (D+1)(1 - \alpha\widetilde{\sigma}) - \alpha\sigma = 0. \tag{50}$$

This has roots at

$$\lambda_+ = \frac{\alpha - (D+2) + \sqrt{(\alpha + D)^2 - 4\alpha D\sigma}}{2} \tag{51}$$

and

$$\lambda_- = \frac{\alpha - (D+2) - \sqrt{(\alpha + D)^2 - 4\alpha D\sigma}}{2}. \tag{52}$$

Since $0 \leq \sigma \leq 1$,

$$(\alpha + D)^2 - 4\alpha D\sigma \geq (\alpha + D)^2 - 4\alpha D = \alpha^2 - 2\alpha D + D^2 = (\alpha - D)^2 \geq 0, \tag{53}$$

and so both eigenvalues are always real. The origin is asymptotically stable if and only if both $\lambda_+ < 0$ and $\lambda_- < 0$.

If $\alpha > D + 2$, then $\lambda_+ > 0$, hence the origin is always unstable in this case. Thus let $\alpha < D + 2$, which covers the special case $\alpha < 1$ (G_1 less advantageous than G_2) but also a wide variety of empirically meaningful cases of $\alpha > 1$. Then $\lambda_- < 0$ always. For λ_+, we find $\lambda_+ < 0$ if and only if

$$\sigma > \frac{(\alpha - 1)(D+1)}{\alpha D} =: \sigma_{\text{crit}}. \tag{54}$$

We have thus found the following necessary and sufficient conditions for the total extinction of the L2-difficult grammar G_1 from both speaker populations:

Proposition 4. *The system* (39) *has a unique (and stable) equilibrium* $(p, q) = (0, 0)$ *if and only if*

$$\alpha < D + 2 \quad \textit{and} \quad \sigma > \sigma_{\text{crit}} = \frac{(\alpha - 1)(D+1)}{\alpha D}. \tag{55}$$

References

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